

Zonal Toroidal
Harmonic Expansion
of External
Gravitational Fields
for Ring-like Objects

aka

Gravitational Field of Baumkuchen



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[ResearchGate Fukushima](#) Click

Submitted to AJ in Revised Form



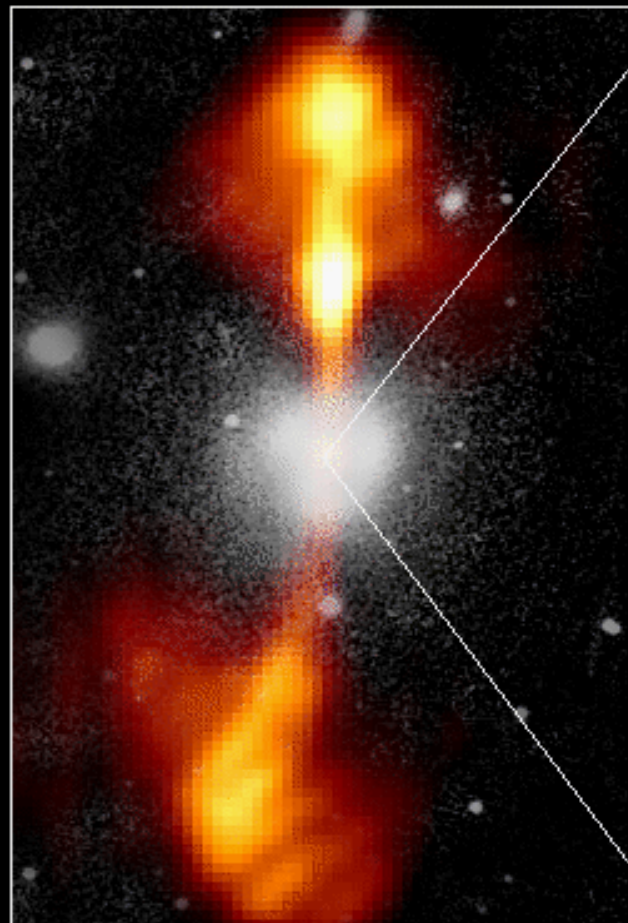
**Baumkuchen
is Not
Only For Party**

Core of Galaxy NGC 4261

Hubble Space Telescope

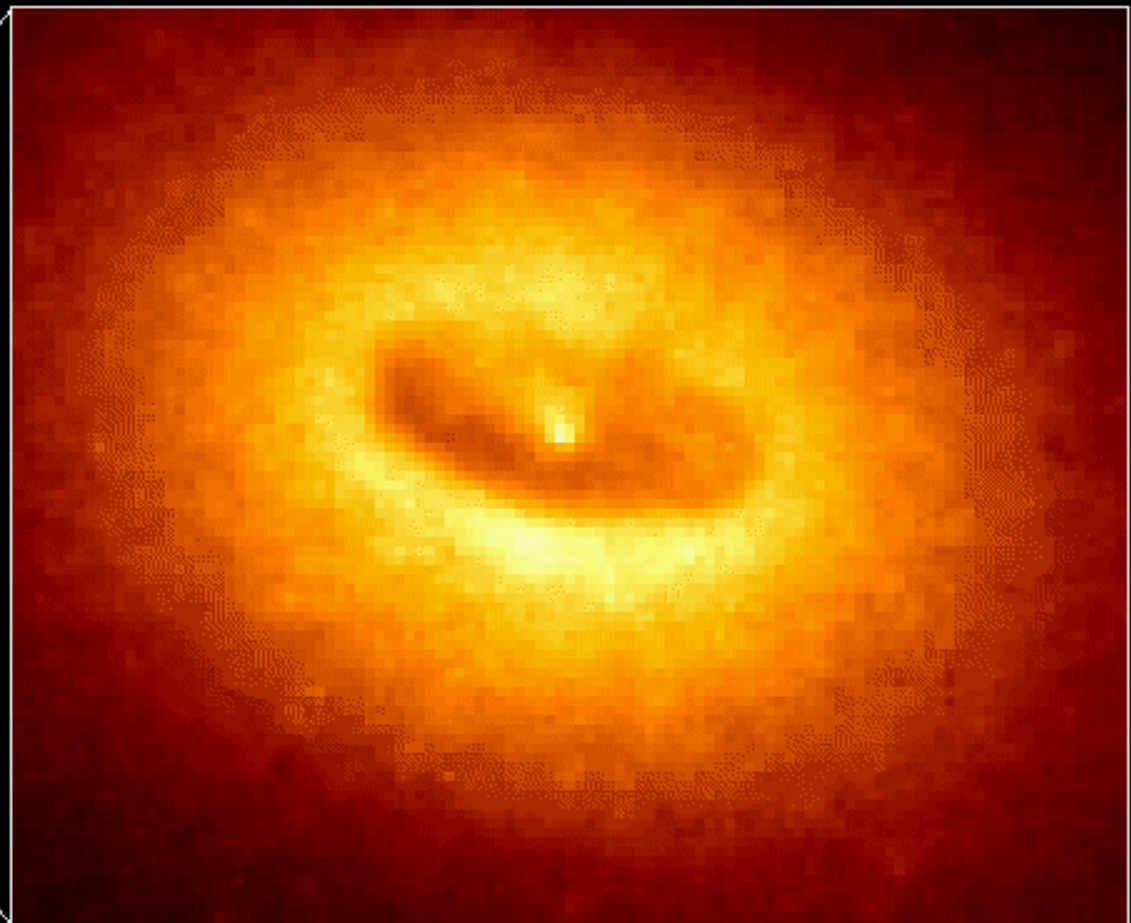
Wide Field / Planetary Camera

Ground-Based Optical/Radio Image



380 Arc Seconds
88,000 LIGHTYEARS

HST Image of a Gas and Dust Disk



17 Arc Seconds
400 LIGHTYEARS

External Grav. Field of Ring-like Object

■ Integral Expression

- Pro: General Purpose
- Con: CPU Time

$$\Phi = \int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$$

■ Harmonic Expansion

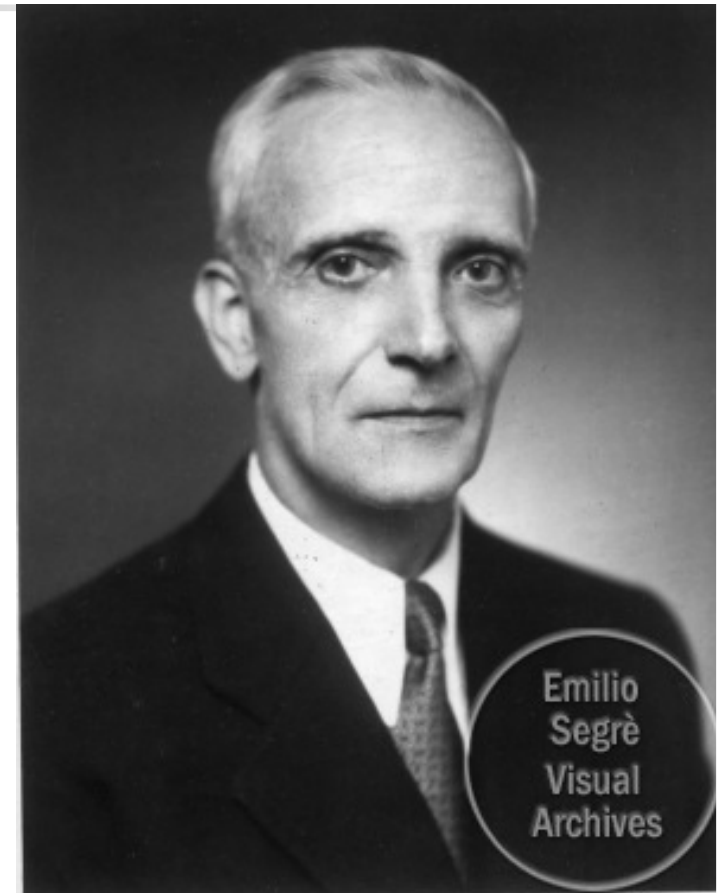
- Pro: Fast Computation
- Con: Lack of Proper Formulation
- WARNING: **Curse** of Brillouin

$$\Phi = \sum_n \Phi_n$$

Curse of Brillouin

Manifold

- Brillouin (1933)
 - Brillouin Scattering, B of WKB
- Brillouin Manifold
= Minimum Circumscribing
Manifold
- **Convergence Limit**
of Harmonic Expansion
- Sphere/Spheroids/Toroid

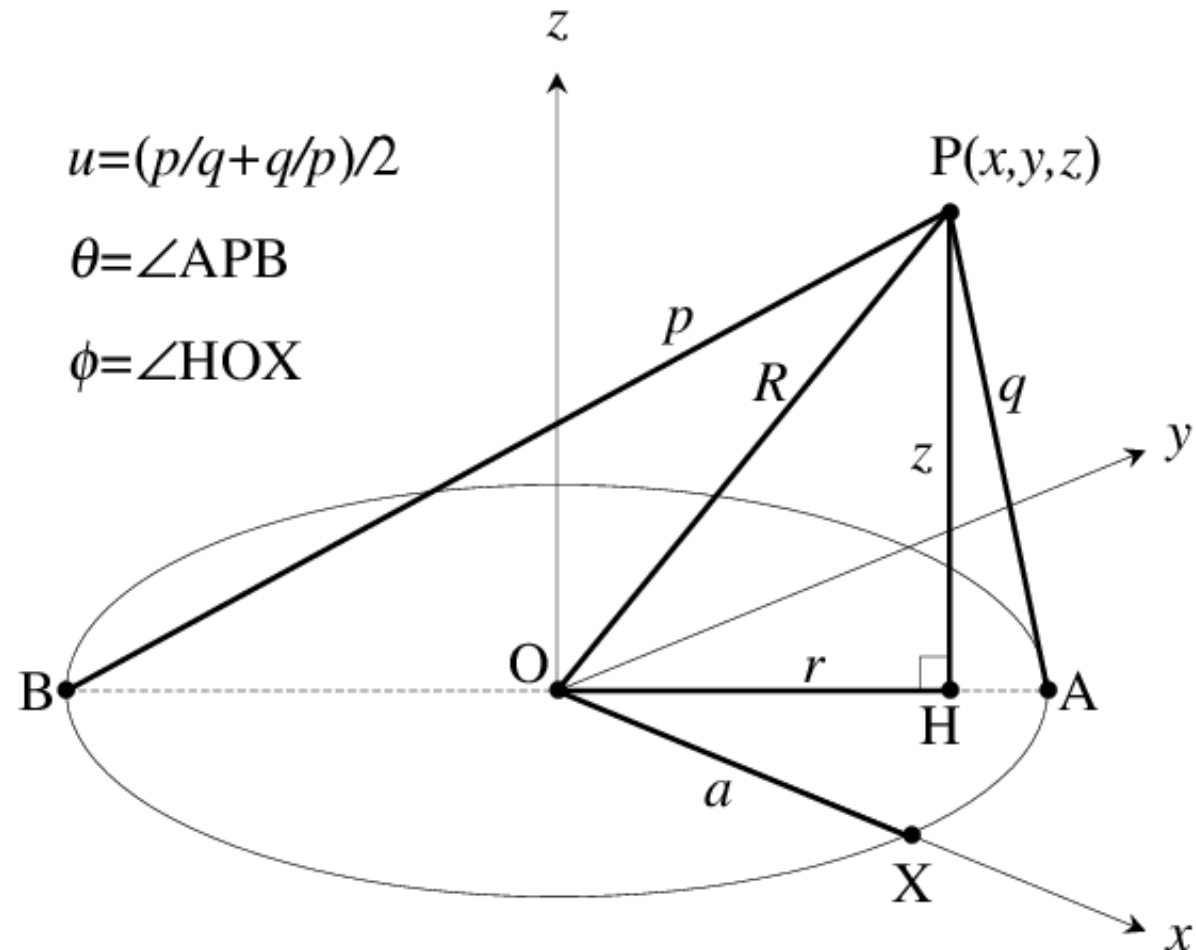


Toroidal Coordinates

 (u, θ, ϕ)

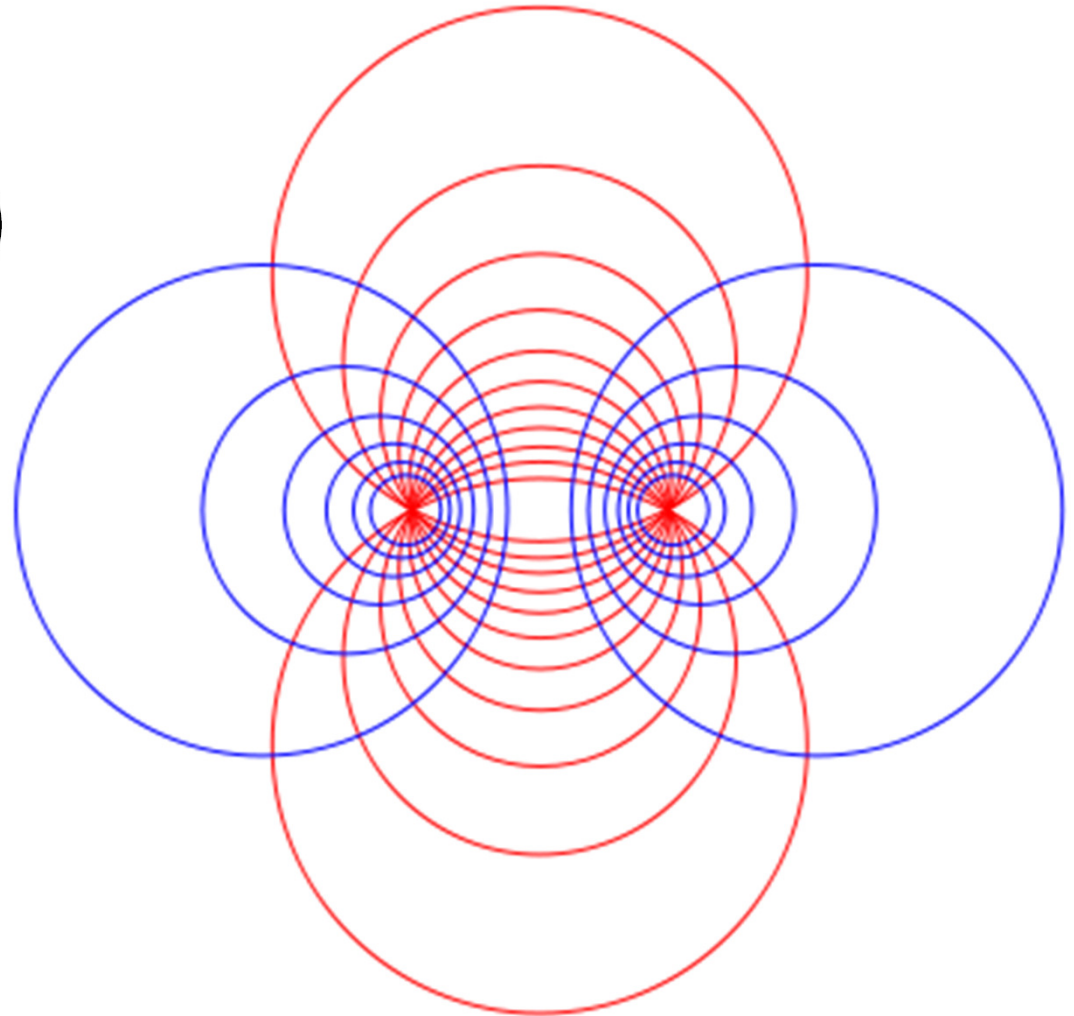
$$r = \frac{a\sqrt{u^2 - 1}}{u - \cos \theta}$$
$$z = \frac{a \sin \theta}{u - \cos \theta}$$

$$x = r \cos \phi$$
$$y = r \sin \phi$$



Toroidal Coordinate Surfaces

- Morse & Feshbach (1953)
- Blue: Constant u
- Red: Constant θ
- Constant Ring Radius means **Different** Center of Toroids



New Theory of Toroidal Harmonic Expansion

- Solution of Laplace's Eq. (Hobson 1931)

$$\Phi = V \sqrt{u - \cos \theta}$$

$$V = \frac{GM}{a} \sum_{n=0}^{\infty} C_n P_n(u) \cos n\theta$$

- **NEW Finding:** Term 0 = Ring Potential at $r=a$
- Key: Fast Computation of Radial Functions

$$p_n(u) \equiv \frac{P_{n-1/2}(u)}{P_{n-1/2}(u_R)} = \left(\frac{u}{u_R} \right)^{n+1/2} \frac{F_n(u)}{F_n(u_R)}$$



Ring Potential

- Classic Result: Ring Radius = a

$$\Phi = \frac{GM}{p} \left(\frac{2K(m)}{\pi} \right)$$

$$m \equiv \frac{4ar}{p^2}$$

$$r \equiv \sqrt{x^2 + y^2}$$

$$p \equiv \sqrt{(r+a)^2 + z^2}$$

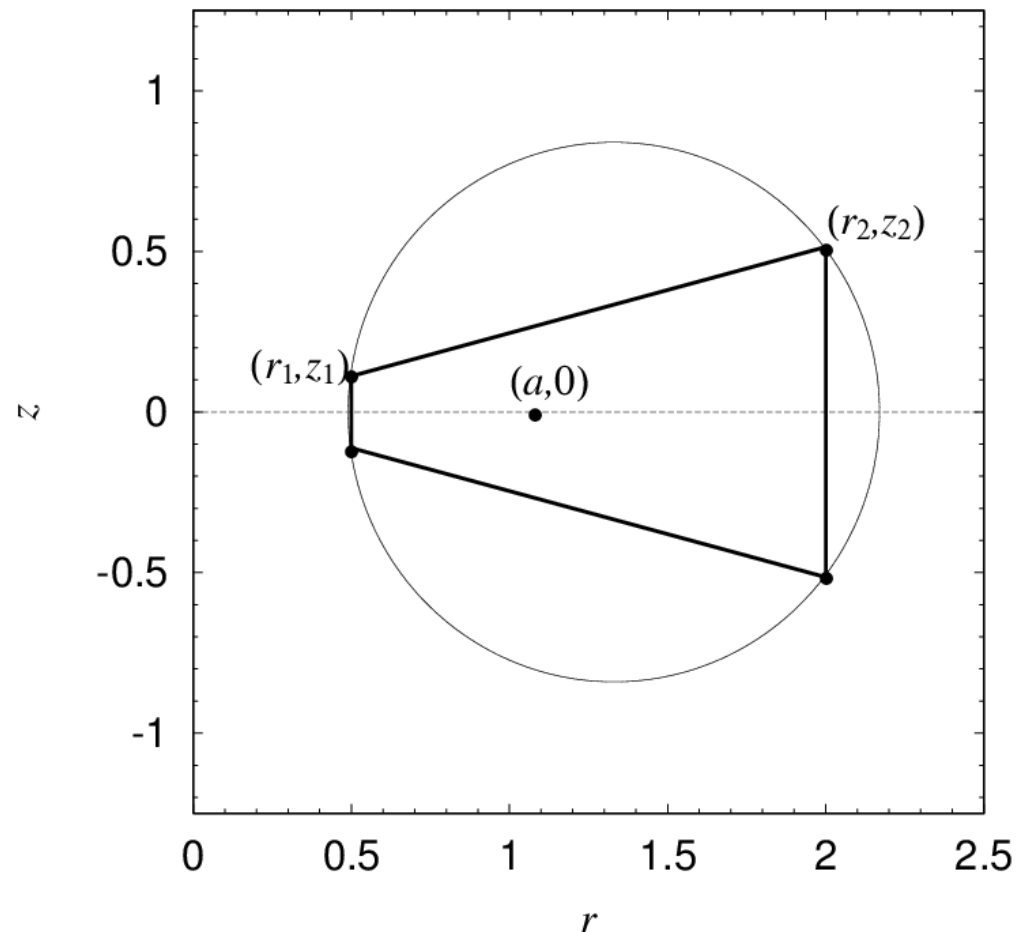
- $K(m)$: Complete Elliptic Integral of 1st Kind
- Acceleration Vector & **Fast** Procedure
 - Fukushima (2010, Cele. Mech. Dyn. Astron.)
 - Fukushima (2015a, J. Comp. Appl. Math.)

**It's Show
Time**

Case of Thick Disk

- $r_0=0.1$
- $r_1=0.5$
- $r_2=2.0$
- $H_0=0.003$
- $\gamma=9/8$
- Surface: 10^{-8}

Thick Disk





Density: Thick Disk

- Axisymmetric Volume Density Profile

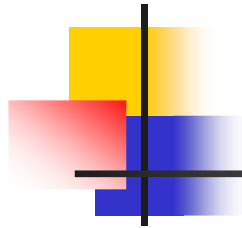
$$\rho_D(r, z) = \rho_0 P(r) \frac{1}{\sqrt{2\pi} H_D(r)} \exp\left[-\frac{1}{2} \left(\frac{z}{H_D(r)}\right)^2\right] \quad r_1 \leq r \leq r_2$$

- Power-Law & Scale Height Functions

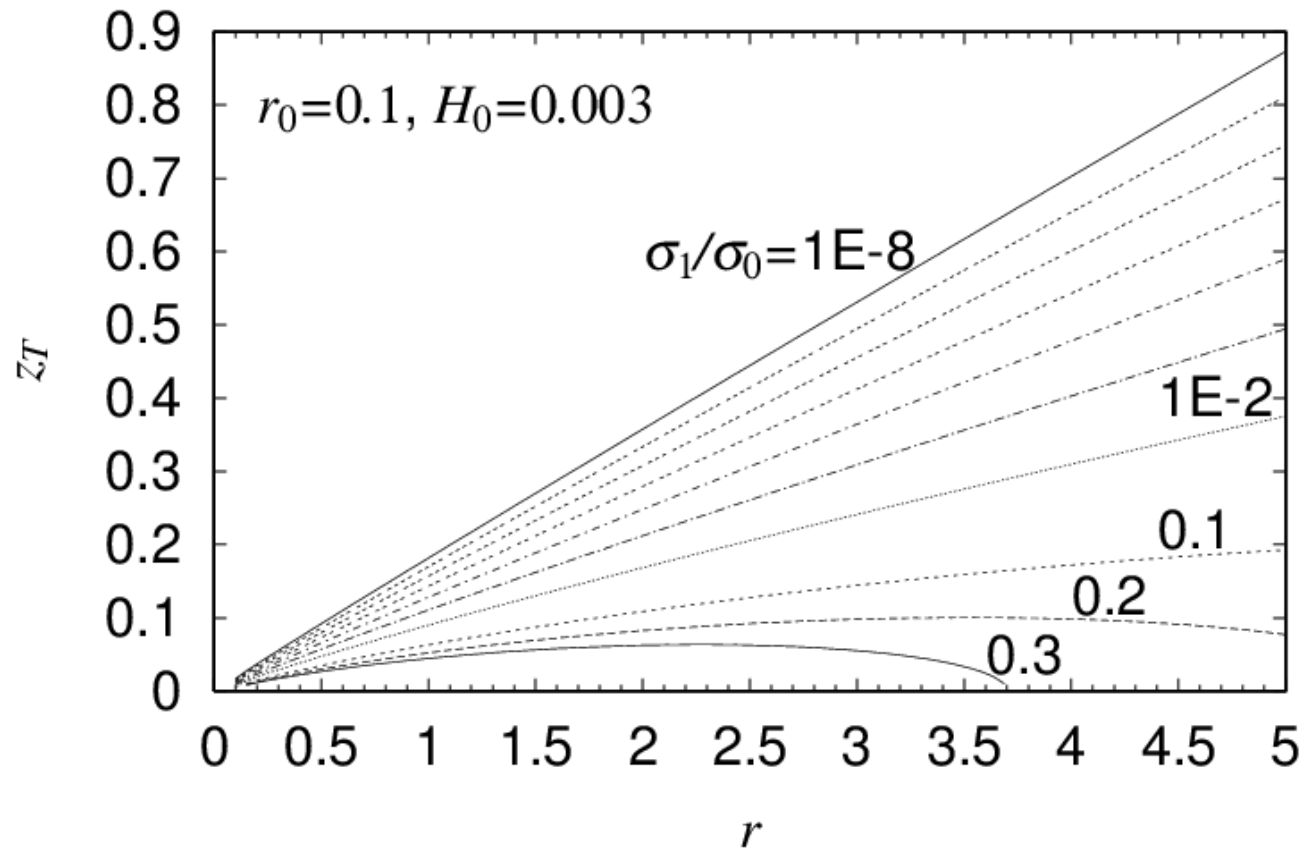
$$P(r) = \left(1 - \sqrt{\frac{r_0}{r}}\right) \frac{r_0}{r}$$

$$H_D(r) = H_0 \left(\frac{r}{r_0}\right)^\gamma$$

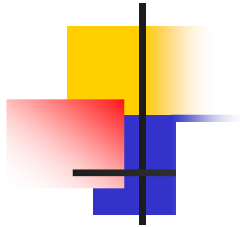
Effective Cross Section of Thick Disk



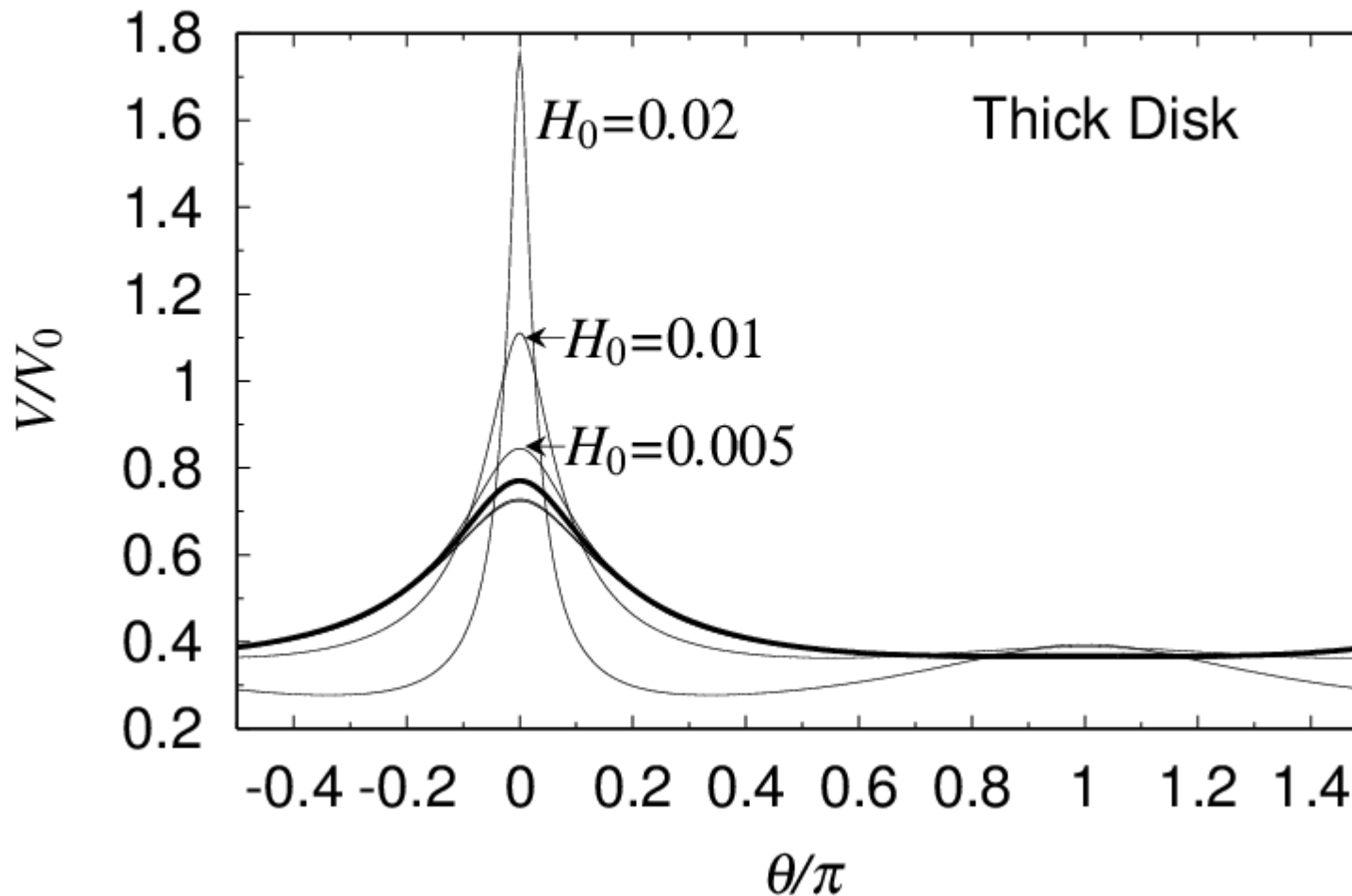
Effective Height $z_T(r)$



Potential: Thick Disk

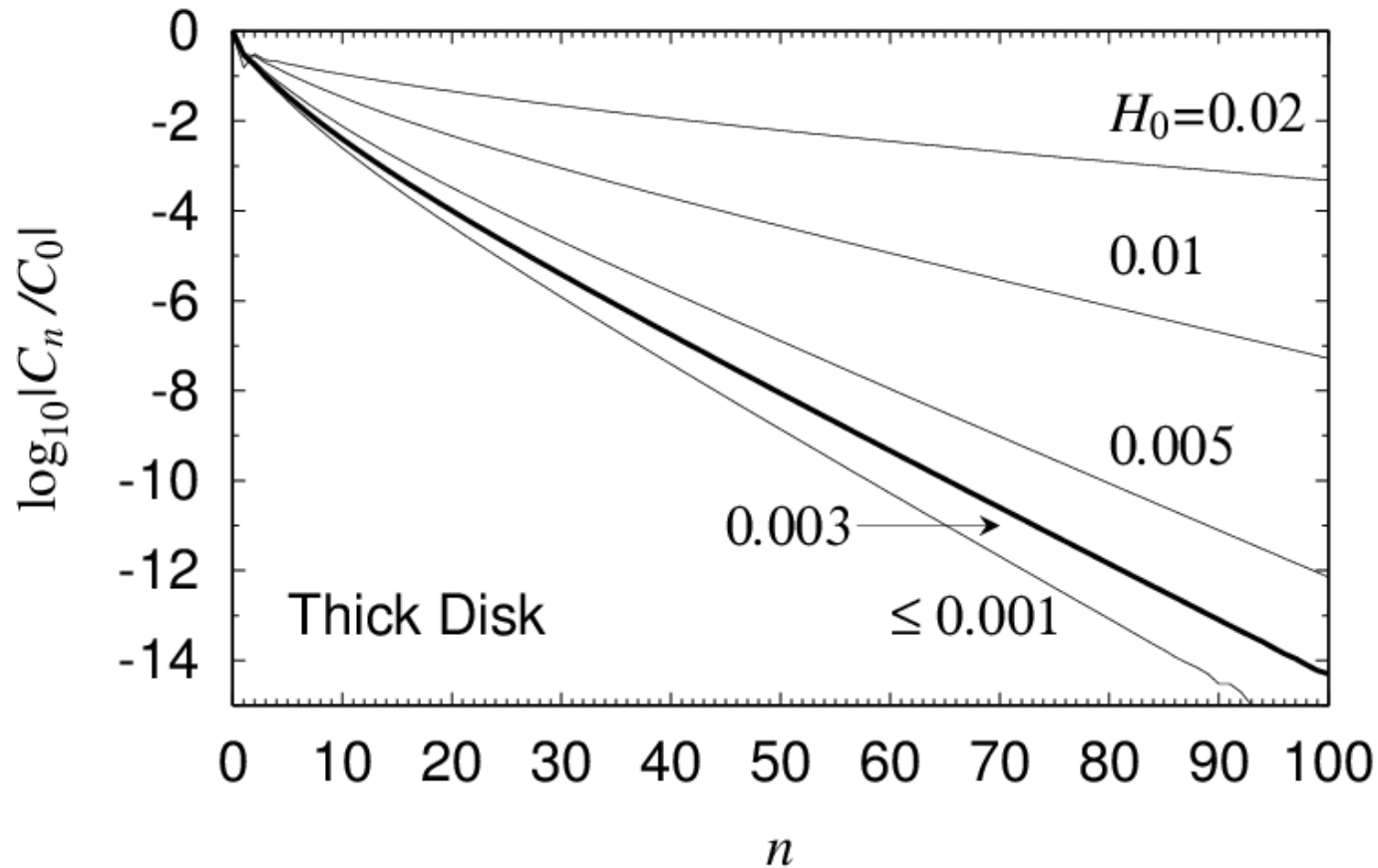


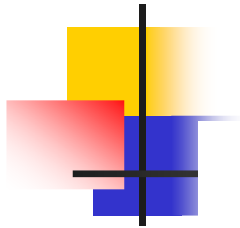
H_0 -Dependence of Reduced Potential



Spectrum: Thick Disk

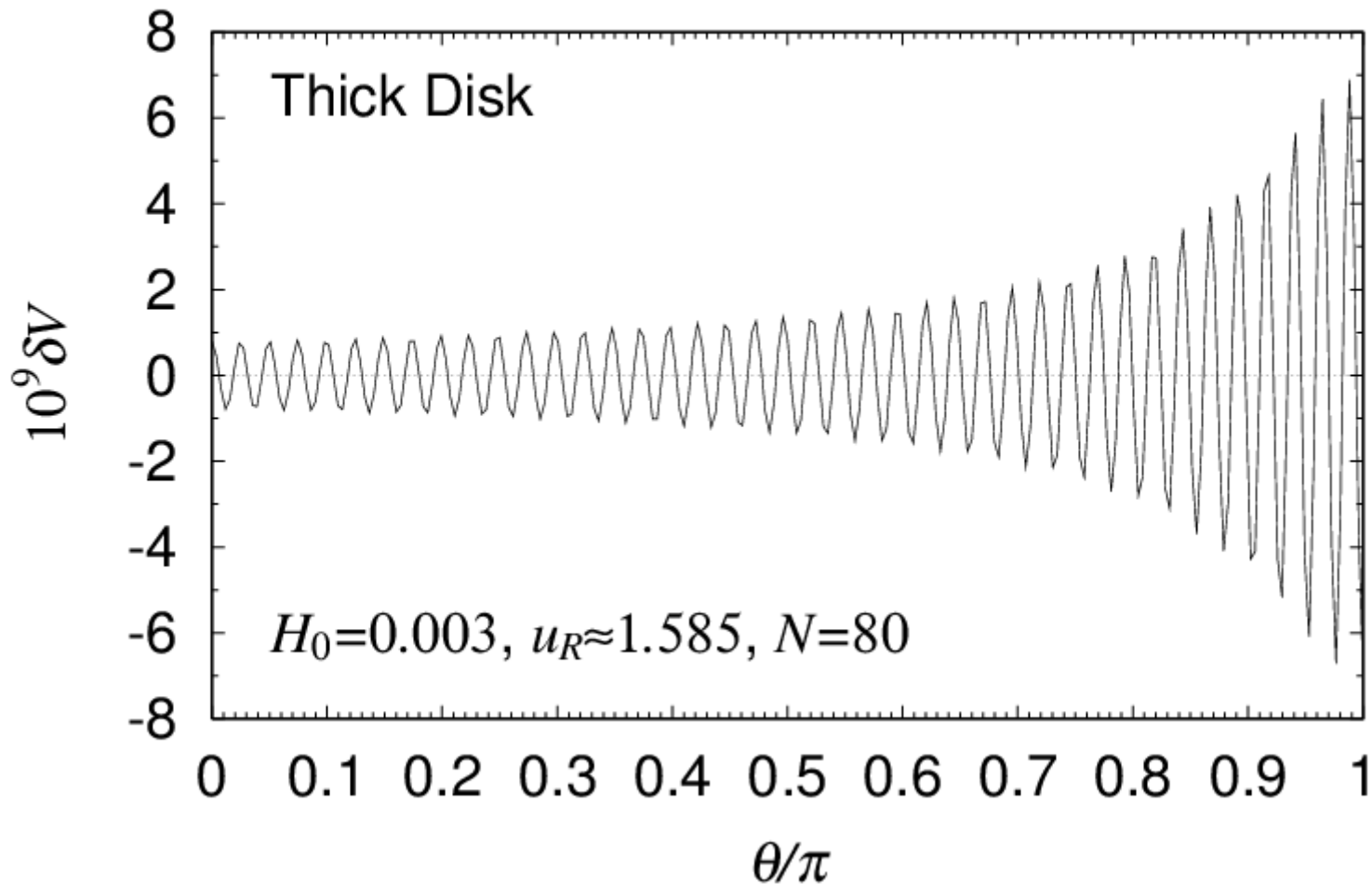
H_0 -Dependence of Spectrum



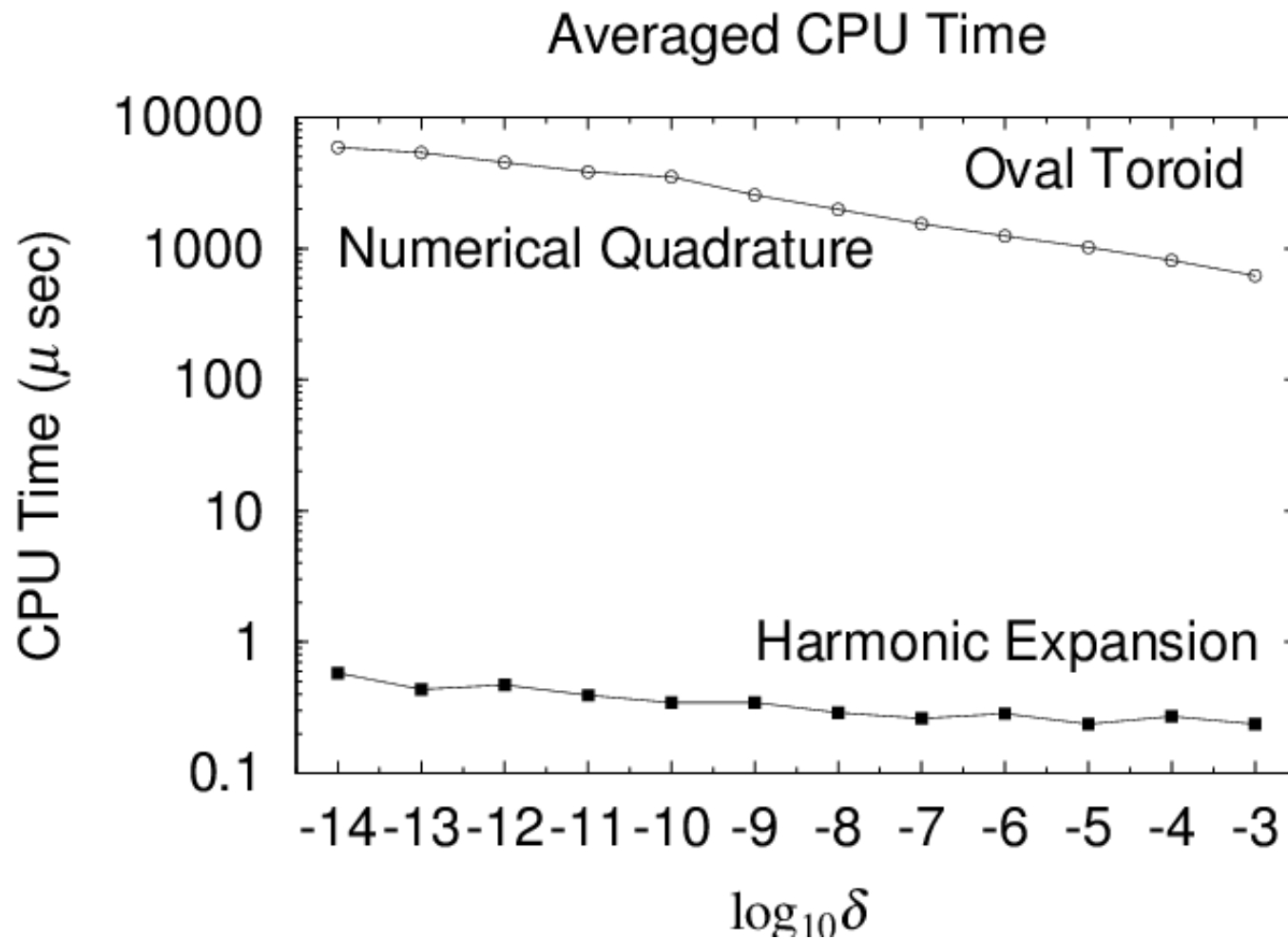


Error Curve: Thick Disk

Relative Error of Truncated Harmonic Expansion



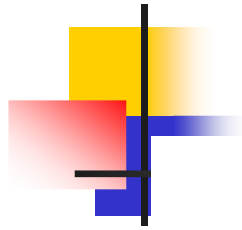
CPU Time Comparison





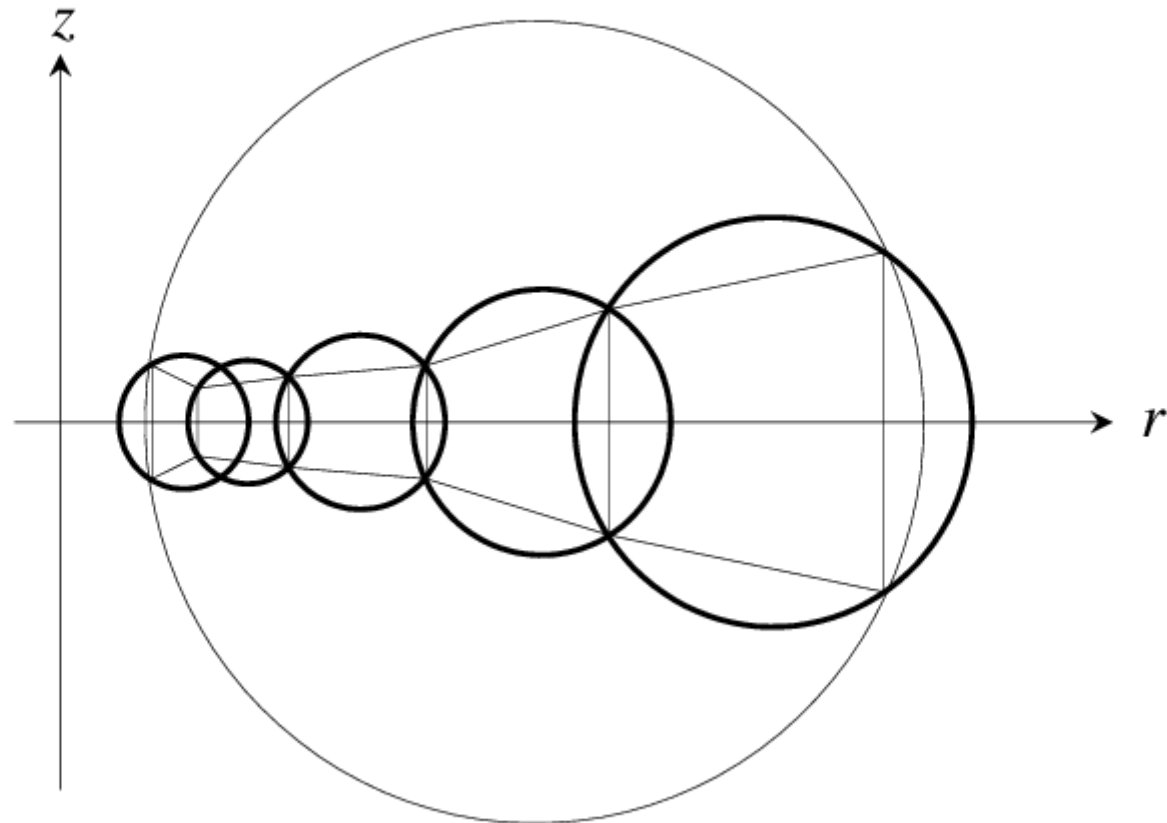
Conclusion

- **New** Theory of Expansion
- Recursive Computation of Radial Func.
- Seeds by Complete Elliptic Integrals
- 3000-10000 Faster Evaluation
- Convergence outside Brillouin Toroid
- Divergence inside Brillouin Toroid
- Accretion Disk \Rightarrow Piecewise Disk Model



Piecewise Disk Model

Piecewise Disk Model





References

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- Fukushima, 2010, Cele. Mech. Dyn. Astron., 108, 339
- Fukushima, 2011, Math. Comp., 80, 1725
- Fukushima, 2015a, J. Comp. Appl. Math., 63, 17
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- Hobson, 1931, The Theory of Spherical and Spheroidal Harmonics, Cambridge Univ. Press
- Morse and Feshbach, 1953, Methods of Theoretical Physics, McGraw-Hill

**Best is
Yet to Come**



Spiral Galaxy M33 (Messier 33)

Subaru Telescope, National Astronomical Observatory of Japan

Suprime-Cam (B, V, H α)
January 22, 2009